Cohomological Methods in Algebraic Combinatorics and Geometry

Alex Abreu

Course Overview

This course explores the interplay between algebraic geometry, representation theory, and combinatorics, focusing on how cohomological tools provide structural understanding and facilitate enumerative computations. We study classical objects such as Schubert varieties, Hessenberg varieties, and moduli spaces of curves, emphasizing their combinatorial invariants.

The course will be held over the months of March, April, and May, consisting of two lectures per week. Each session will last two hours, for a total of 52 hours.

Course Schedule

Week 1: Foundations of Cohomology in Algebraic Geometry

Lecture 1: Introduction to cohomology and Chow groups. Intersection theory basics.

Lecture 2: Cup product, pushforward/pullback, Poincaré duality. Examples in projective spaces and Grassmannians.

Week 2: Equivariant Cohomology and GKM Theory

Lecture 3: Equivariant cohomology: Borel construction, basic computations.

Lecture 4: GKM theory: definition, GKM varieties, combinatorial description via moment graphs.

Weeks 3-4: Flag Varieties and Schubert Calculus

Lecture 5: Flag varieties: geometry, Bruhat decomposition, Schubert cells.

Lecture 6: Schubert classes, basis for cohomology, and ring structure.

Lecture 7: Kazhdan-Lusztig theory: Hecke algebras, KL polynomials, intersection cohomology.

Lecture 8: Cohomology of Schubert varieties: singularities, resolutions.

Weeks 5-6: Perverse Sheaves and Local Systems

Lecture 9: Sheaves, local systems, Verdier duality.

Lecture 10: Overview on Perverse sheaves and the decomposition theorem.

Lecture 11: Applications to intersection cohomology and Kazhdan-Lusztig theory.

Lecture 12: Further applications: examples from flag varieties.

Week 7: Symmetric Functions and Chromatic Symmetric Functions

Lecture 13: Symmetric functions and representation theory of S_n: Schur functions, Specht modules, Frobenius characteristic.

Lecture 14: Chromatic symmetric functions: definition, properties, positivity conjectures, connections to geometry.

Weeks 8-9: Hessenberg and Lusztig Varieties

Lecture 15: Hessenberg varieties: definition, examples, cell decompositions.

Lecture 16: Lusztig varieties, parabolic generalizations, relation to Springer fibers.

Lecture 17: Cohomology of Hessenberg varieties: connections to chromatic symmetric functions.

Lecture 18: Shareshian-Wachs conjecture, GKM theory applications in Hessenberg varieties.

Weeks 10-11: Tautological Rings in Moduli Spaces

Lecture 19: Moduli spaces of curves $\overline{\mathcal{M}}_{g,n}$: geometry, basics of cohomology and intersection theory.

Lecture 20: Tautological ring: definitions, generators, relations, Mumford's formula, Faber's conjectures.

Lecture 21: Tautological rings of toroidal schemes: relation to intersection theory.

Lecture 22: Equivariant-like descriptions of tautological rings, connections to GKM theory.

Weeks 12-13: Tautological Ring of $\overline{\mathcal{M}}_{g,n}$ – Computations

Lecture 23: Locus of Weierstrass points.

Lecture 24: Double Ramification Cycle.

Lectures 25-26: More Examples.

References

[1] I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd edition.

[2] W. Fulton, *Intersection Theory*, 2nd edition.

[3] E. Arbarello, M. Cornalba, P. Griffiths, J. Harris, *Geometry of Algebraic Curves*, Vol. II.